

Exact multiplicity of solutions for some semilinear Dirichlet problems

Philip Korman
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025

Abstract

The classical result of A. Ambrosetti and G. Prodi [1], in the form of M.S. Berger and E. Podolak [4], gives the exact number of solutions for the problem

$$\Delta u + g(u) = \mu \phi_1(x) + e(x) \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

depending on the real parameter μ , for a class of convex $g(u)$, and $\int_D e(x) \phi_1(x) dx = 0$ (where $\phi_1(x) > 0$ is the principal eigenfunction of the Laplacian on D , and $D \subset \mathbb{R}^n$ is a smooth domain). By considering generalized harmonics, we give a similar result for the problem

$$\Delta u + g(u) = \mu f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

with $f(x) > 0$. Such problems occur, for example, in “fishing” applications that we discuss, and propose a new model.

Our approach also produces a very simple proof of the anti-maximum principle of Ph. Clément and L.A. Peletier [5].

Key words: Global solution curves, exact number of solutions, the anti-maximum principle.

AMS subject classification: 35J61, 35J25, 92D25.

1 Introduction

Consider the problem

$$(1.1) \quad \Delta u + g(u) = f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

where D is a smooth domain in R^n , and the functions $g(u)$ and $f(x)$ are given. Decompose $f(x) = \mu\phi_1(x) + e(x)$, where $\phi_1(x) > 0$ is the principal eigenfunction of the Laplacian on D with zero boundary condition, and $\int_D e(x)\phi_1(x) dx = 0$. The classical result of A. Ambrosetti and G. Prodi [1], in the form of M.S. Berger and E. Podolak [4], says that if $g(u)$ is convex and asymptotically linear at $\pm\infty$, then (under an additional restriction on the slopes of $g(u)$ at $\pm\infty$) there exists a critical $\mu_0 = \mu_0(e(x))$, such that the problem (1.1) has exactly two solutions for $\mu > \mu_0$, exactly one solution if $\mu = \mu_0$, and no solutions for $\mu < \mu_0$. However, sometimes it is desirable to have the parameter μ in front of the entire right hand side, and to consider the problem

$$(1.2) \quad \Delta u + g(u) = \mu f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

Such problems occur e.g., when one considers “fishing” applications, see S. Oruganti et al [15], D.G. Costa et al [6], P. Girão, and H. Tehrani [8], P.M. Girão and M. Pérez-Llanos [7]. We present an exact multiplicity result of Berger-Podolak type for the problem (1.2), provided that $f(x) > 0$ on D . Throughout the paper, one can easily replace the Laplacian by any uniformly elliptic operator.

Similar result holds for the problem

$$\Delta u + g(u) = \mu f(x) + e(x) \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

with $f(x) > 0$ on D , and $\int_D e(x)f(x) dx = 0$, providing a generalization of the above mentioned result of M.S. Berger and E. Podolak [4].

Our approach involves applying the implicit function theorem for continuation of solutions in a special way. We restrict the space of solutions by keeping the generalized first harmonic fixed, but in return allow μ to vary. Then we compute the direction of the turn of the solution curve, similarly to P. Korman [9]. We show that there is at most one turn in case $g(u)$ is either convex or concave.

The well-known anti-maximum principle of Ph. Clément and L.A. Peletier [5] follows easily with this approach.

We apply our results to a population model with fishing. We suggest a modification of the logistic model, to admit sign-changing solutions. We argue that one needs to consider sign-changing solutions to get complete bifurcation diagrams.

2 The global solution curves

We assume that D is a smooth domain in R^n , and denote by λ_k the eigenvalues of the Laplacian on D , with zero boundary conditions, and by $\varphi_k(x)$ the corresponding eigenfunctions, normalized so that $\int_D \varphi_k^2(x) dx = 1$. It is known that $\varphi_1(x) > 0$ is simple, and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$. We denote by $H^k(D)$ the Sobolev spaces $W^{k,2}(D)$. We shall need the following generalization of Poincare's inequality.

Lemma 2.1 *Let $u(x) \in H_0^1(D)$ be such that $\int_D u(x)f(x) dx = 0$, for some $f(x) \in L^2(D)$, $f(x) \not\equiv 0$. Denote $f_1 = \int_D f(x)\varphi_1(x) dx$, and*

$$\nu = \lambda_1 + (\lambda_2 - \lambda_1) \frac{f_1^2}{\|f\|_{L^2}^2}.$$

Then

$$(2.1) \quad \int_D |\nabla u|^2 dx \geq \nu \int_D u^2 dx.$$

Proof: By scaling of $u(x)$, we may assume that $\int_D u^2(x) dx = 1$. Writing

$$u(x) = \sum_{k=1}^{\infty} u_k \varphi_k(x), \quad f(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x),$$

we then have

$$(2.2) \quad \sum_{k=1}^{\infty} u_k^2 = 1,$$

$$(2.3) \quad \sum_{k=1}^{\infty} u_k f_k = 0.$$

We need to show that $\int_D |\nabla u|^2 dx \geq \nu$. Using (2.2), we estimate

$$(2.4) \quad \begin{aligned} \int_D |\nabla u|^2 dx &= - \int_D u \Delta u dx = \sum_{k=1}^{\infty} \lambda_k u_k^2 \\ &= \lambda_1 + \sum_{k=2}^{\infty} (\lambda_k - \lambda_1) u_k^2 \geq \lambda_1 + (\lambda_2 - \lambda_1) \sum_{k=2}^{\infty} u_k^2. \end{aligned}$$

From (2.3)

$$(2.5) \quad \left(\sum_{k=2}^{\infty} u_k^2 \right)^{1/2} \left(\sum_{k=2}^{\infty} f_k^2 \right)^{1/2} \geq \sum_{k=2}^{\infty} u_k f_k = -u_1 f_1 = |u_1| |f_1|.$$

Set $x = (\sum_{k=2}^{\infty} u_k^2)^{1/2}$, $f = (\sum_{k=2}^{\infty} f_k^2)^{1/2}$. In view of (2.2), we get from (2.5)

$$xf \geq |f_1| \sqrt{1 - x^2},$$

or

$$x^2 \geq \frac{f_1^2}{f^2 + f_1^2} = \frac{f_1^2}{\|f\|_{L^2}^2},$$

and the proof follows from (2.4). \diamond

The inequality (2.1) is sharp in the following sense: when $f = \varphi_1$, we have $\nu = \lambda_2$, and one has an equal sign in (2.1) at $u = \varphi_2$. Clearly, $\nu \leq \lambda_2$, and $\nu > \lambda_1$ if $f(x) > 0$.

Lemma 2.2 *Let $(w(x), \mu) \in H^2(D) \times R$ solve the problem*

$$(2.6) \quad \begin{aligned} \Delta w + a(x)w &= \mu f(x) \text{ in } D, \quad w = 0 \text{ on } \partial D \\ \int_D w(x)f(x) dx &= 0, \end{aligned}$$

with some $f(x) \in L^2(D)$, $f(x) \not\equiv 0$. Assume that $a(x) \in C(D)$ satisfies $a(x) < \nu$ for all $x \in D$. Then $w(x) \equiv 0$, and $\mu = 0$.

Proof: Multiply the equation in (2.6) by w , and integrate. By Lemma 2.1, we have

$$\nu \int_D w^2 dx \leq \int_D |\nabla w|^2 dx = \int_D a(x)w^2 dx < \nu \int_D w^2 dx.$$

Hence, $w(x) \equiv 0$, and from (2.6), $\mu = 0$. \diamond

Lemma 2.3 *Consider the problem (to find $z(x)$ and μ^*)*

$$(2.7) \quad \begin{aligned} \Delta z + a(x)z &= \mu^* f(x) + e(x) \text{ in } D, \quad w = 0 \text{ on } \partial D \\ \int_D z(x)f(x) dx &= \xi, \end{aligned}$$

where $f(x) \in L^2(D)$ satisfies $f(x) > 0$ a.e., and $a(x) \in C(D)$ satisfies $a(x) < \nu$ for all $x \in D$. Then for any $e(x) \in L^2(D)$, and any $\xi \in R$, the problem has a solution $(z(x), \mu^) \in (H^2(D) \cap H_0^1(D)) \times R$.*

Proof: *Case 1.* Assume that the operator

$$L[z] \equiv \Delta z + a(x)z : H^2(D) \cap H_0^1(D) \rightarrow L^2(D)$$

is invertible. We claim that

$$(2.8) \quad \int_D L^{-1}(f(x))f(x) dx \neq 0,$$

where L^{-1} denotes the the inverse operator of $L[z]$. Indeed, assuming otherwise, $w(x) \equiv L^{-1}(f(x))$ is not identically zero, and it satisfies (2.6), with $\mu = 1$, which contradicts Lemma 2.2. Then the solution of (2.7) is

$$z(x) = \mu^* L^{-1}(f(x)) + L^{-1}(e(x)),$$

and μ^* is chosen so that $\int_D z(x)f(x) dx = \xi$, which we can accomplish, in view of (2.8).

Case 2. Assume that the operator $L[z]$ is not invertible. Since $a(x) < \nu \leq \lambda_2$, the kernel of $L[z]$ is one-dimensional, spanned by some $\varphi(x) > 0$. Since $L[z]$ is a Fredholm operator of index zero, the first equation in (2.7) is solvable if and only if its right hand side is orthogonal to $\varphi(x)$. We now obtain the solution $(z(x), \mu^*)$ of (2.7) as follows. Choose μ^* , so that $\int_D (\mu^* f(x) + e(x)) \varphi(x) dx = 0$. Then the first equation in (2.7) has infinitely many solutions of the form

$$z(x) = z_0(x) + c\varphi(x),$$

with some $z_0(x)$. We choose the constant c , so that $\int_D z(x)f(x) dx = \xi$. \diamond

We consider next the nonlinear problem

$$(2.9) \quad \Delta u + g(u) = \mu f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

We shall assume that $g(u) \in C^1(R)$, and

$$(2.10) \quad g(u) = \begin{cases} \gamma_1 u + b_1(u) & \text{if } u < 0 \\ \gamma_2 u + b_2(u) & \text{if } u \geq 0, \end{cases}$$

with real constants γ_1, γ_2 , and $b_1(u), b_2(u)$ bounded for all $u \in R$. Notice that we admit the case of $\gamma_2 = \gamma_1$, and in particular we allow bounded $g(u)$, in case $\gamma_2 = \gamma_1 = 0$. We shall consider strong solutions of (2.9), $u(x) \in H^2(D) \cap H_0^1(D)$.

Any function $u(x) \in L^2(D)$ can be decomposed as

$$(2.11) \quad u(x) = \xi f(x) + U(x), \quad \text{with } \int_D U(x)f(x) dx = 0,$$

for any $f(x) \in L^2(D)$. If $f(x) > 0$ a.e., we call the constant ξ the *generalized first harmonic* of $u(x)$.

We shall need the following a priori estimate.

Lemma 2.4 Assume that $f(x) \in H^2(D)$, $f(x) > 0$ a.e., and $g(u) \in C^1(R)$ satisfies the condition (2.10), and $g'(u) \leq \nu_1$, for some constant $\nu_1 < \nu$. Let $u(x) \in H^2(D) \cap H_0^1(D)$ be a solution of (2.9), decomposed as in (2.11). Then for some positive constants c_1 and c_2

$$(2.12) \quad |\mu| + \|U\|_{H^2(D)} \leq c_1 |\xi| + c_2.$$

Proof: Using the ansatz (2.11) in (2.9), we have

$$(2.13) \quad \Delta U + \xi \Delta f + g(\xi f(x) + U) = \mu f(x) \text{ in } D, \quad U = 0 \text{ on } \partial D.$$

Multiplying by U and integrating, we write the result as

$$(2.14) \quad \int_D |\nabla U|^2 dx - \int_D (\xi \Delta f) U dx - \int_D [g(\xi f(x) + U) - g(\xi f(x))] U dx - \int_D g(\xi f(x)) U dx = 0.$$

Using the mean value theorem, we estimate from below the third term on the left by $-\nu_1 \int_D U^2 dx$. If $\xi \geq 0$, then

$$\int_D g(\xi f(x)) U dx = \int_D (\gamma_2 \xi f(x) + b_2(\xi f(x))) U dx = \int_D b_2(\xi f(x)) U dx.$$

Using Lemma 2.1, we have from (2.14), for any small $\epsilon > 0$,

$$\begin{aligned} (\nu - \nu_1) \int_D U^2 dx &\leq \int_D (\xi \Delta f) U dx + \int_D b_2(\xi f(x)) U dx \\ &\leq \epsilon \int_D U^2 dx + c(\epsilon) \xi^2 \int_D (\Delta f)^2 dx + \epsilon \int_D U^2 dx + c(\epsilon), \end{aligned}$$

which gives us an estimate of $\int_D U^2 dx$

$$\int_D U^2 dx \leq c_1 \xi^2 + c_2, \quad \text{uniformly in } \mu,$$

with some positive constants c_1, c_2 . In case $\xi < 0$, the same estimate follows similarly. Returning to (2.14), and using (2.10), we have

$$(2.15) \quad \int_D (|\nabla U|^2 + U^2) dx \leq c_1 \xi^2 + c_2, \quad \text{uniformly in } \mu.$$

(Here and later on, c_1, c_2 denote possibly new positive constants.) Then

$$(2.16) \quad \int_D (|\nabla u|^2 + u^2) dx \leq c_1 \xi^2 + c_2, \quad \text{uniformly in } \mu.$$

To get an estimate of μ , we now multiply (2.9) by $u = \xi f + U$, and integrate

$$\xi \mu \int_D f^2 dx = - \int_D |\nabla u|^2 dx + \int_D g(u) u dx,$$

which in view of (2.16) implies that

$$(2.17) \quad |\xi| |\mu| \leq c_1 \xi^2 + c_2.$$

(Observe that $|g(u)| \leq A|u| + B$, for some positive constants A, B , and for all u .) Fix some $\xi_0 > 0$. Then for $|\xi| \geq \xi_0$, we conclude from (2.17)

$$(2.18) \quad |\mu| \leq c_1 |\xi| + c_2.$$

In case $|\xi| \leq \xi_0$, we multiply (2.9) by φ_1 , and integrate to show that $|\mu| \leq c_3$, for some $c_3 > 0$. We conclude that the bound (2.18) holds for all $\xi \in R$.

We multiply (2.9) by Δu , and integrate. Obtain

$$\int_D (\Delta u)^2 dx + \int_D \Delta u g(u) dx = -\mu \int_D \nabla f \cdot \nabla u dx.$$

Using the estimates (2.16) and (2.18), we get

$$\int_D (\Delta u)^2 dx \leq c_1 \xi^2 + c_2.$$

Since $\Delta u = \Delta U + \xi \Delta f$, we conclude that

$$\int_D (\Delta U)^2 dx \leq c_1 \xi^2 + c_2.$$

By the elliptic estimates we obtain the desired bound on $\|U\|_{H^2(D)}$. \diamond

Corollary 1 *In case $f(x) = \varphi_1(x)$, the second term on the left in (2.14) vanishes, and we conclude that*

$$\|U\|_{H^1(D)} \leq c, \quad \text{uniformly in } \xi \text{ and } \mu,$$

for some constant $c > 0$.

Theorem 2.1 *Assume that $f(x) \in H^2(D)$, $f(x) > 0$ a.e., and $g(u) \in C^1(R)$ satisfies the condition (2.10), and we have $g'(u) \leq \nu_1 < \nu$ for all $u \in R$. Then for each $\xi \in (-\infty, \infty)$, there exists a unique μ , for which the problem (2.9) has a unique solution $u(x) \in H^2(D) \cap H_0^1(D)$, with the generalized first harmonic equal to ξ . The function $\mu = \phi(\xi)$ is smooth.*

Proof: We embed (2.9) into a family of problems

$$(2.19) \quad \Delta u + \lambda_1 u + k(g(u) - \lambda_1 u) - \mu f(x) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

with $0 \leq k \leq 1$ ($k = 1$ corresponds to (2.9)). When $(k = 0, \mu = 0)$ the problem has solutions $u = a\varphi_1$, where a is any constant. By choosing $a = a_0$, we can get the solution $u = a_0\varphi_1$ of any generalized first harmonic ξ^0 . We now continue in k the solutions of

$$(2.20) \quad F(u, \mu, k) \equiv \Delta u + \lambda_1 u + k(g(u) - \lambda_1 u) - \mu f(x) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D$$

$$\int_D u f dx = \xi^0,$$

with the operator $F(u, \mu, k) : H^2(D) \times R \times R \rightarrow L^2(D)$. We will show that the implicit function theorem applies, allowing us to continue (u, μ) as a function of k . Compute the Frechet derivative

$$F_{(u, \mu)}(u, \mu, k)(w, \mu^*) = \Delta w + \lambda_1 w + k(g'(u) - \lambda_1)w - \mu^* f(x), \\ \int_D w f dx = 0.$$

By Lemma 2.2, the map $F_{(u, \mu)}(u, \mu, k)(w, \mu^*)$ is injective, and by Lemma 2.3 this map is surjective. Hence, the implicit function theorem applies, and we have a solution curve $(u, \mu)(k)$. By the a priori estimate of Lemma 2.4, this curve continues for all $0 \leq k \leq 1$, and at $k = 1$, we obtain a solution of the problem (2.9) with the generalized first harmonic equal to ξ^0 .

Turning to the uniqueness, let $(\bar{\mu}, \bar{u}(x))$ be another solution of (2.9), and $\bar{u}(x)$ has the generalized first harmonic equal to ξ^0 . Then $(\bar{\mu}, \bar{u}(x))$ is solution of (2.20) at $k = 1$. We continue this solution backward in k , until $k = 0$, using the implicit function theorem. By the Fredholm alternative, we have $\mu = 0$, when $k = 0$. Then $u = a_1\varphi_1$, with $a_1 \neq a_0$ (since the solution curves do not intersect), and $\bar{u}(x)$ has the generalized first harmonic equal to ξ^0 , a contradiction.

Finally, we show that solutions of (2.9) can be continued in ξ , by using the implicit function theorem. Decomposing $u(x) = \xi f(x) + U(x)$, with $\int_D U f dx = 0$, we see that $U(x)$ satisfies

$$F(U, \mu, \xi) \equiv \Delta U + g(\xi f(x) + U(x)) - \mu f(x) = 0 \text{ in } D, \quad U = 0 \text{ on } \partial D \\ \int_D U f dx = 0.$$

Compute the Frechet derivative

$$F_{(U,\mu)}(U, \mu, \xi)(w, \mu^*) = \Delta w + g'(\xi f(x) + U(x)) w - \mu^* f(x),$$

$$\int_D w f dx = 0.$$

As before, we see that the implicit function theorem applies, and we have a smooth solution curve $(u, \mu)(\xi)$ for the problem (2.9). By Lemma 2.4, this curve continues for all $\xi \in R$. \diamond

Remark The theorem implies that the value of ξ is a *global parameter*, uniquely identifying the solution pair $(\mu, u(x))$.

The well-known anti-maximum principle is easily proved by a similar argument. As in J. Shi [17], we state it along with the classical maximum principle. We present a self-contained proof, since the a priori estimate of Lemma 2.4 is not needed for this local result.

Theorem 2.2 *Consider the following problem, with $f(x) \in L^2(D)$, and $f(x) > 0$ a.e. in D ,*

$$(2.21) \quad \Delta u + \lambda u = f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

Then there exists a constant δ_f , which depends on f , such that if $\lambda_1 < \lambda < \lambda_1 + \delta_f$, then

$$(2.22) \quad u(x) > 0, \quad x \in D, \quad \frac{\partial u}{\partial n} < 0, \quad x \in \partial D;$$

and if $\lambda < \lambda_1$, then

$$u(x) < 0, \quad x \in D, \quad \frac{\partial u}{\partial n} > 0, \quad x \in \partial D.$$

Proof: We prove the first part. Consider the problem

$$(2.23) \quad \Delta u + \lambda_1 u + k u = \mu f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

When $k = 0$, and $\mu = 0$, this problem has a solution $u = \varphi_1$. Decompose $\varphi_1 = \xi^0 f(x) + e(x)$, with $\int_D f(x) e(x) dx = 0$. We now continue in k , $k \geq 0$ the solution $(u, \mu) \in (H^2(D) \cap H_0^1(D)) \times R$ of

$$\Delta u + \lambda_1 u + k u = \mu f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D$$

$$\int_D u f dx = \xi^0,$$

beginning with $(\varphi_1, 0)$ at $k = 0$. By Lemmas 2.2 and 2.3, the implicit function theorem applies, and we have a solution curve $(u, \mu)(k)$, at least

for small k . If $k > 0$ is small, then $u(x)$ is close to $\varphi_1(x)$, and we have $u(x) > 0$ in D , and $\frac{\partial u}{\partial n} < 0$ on ∂D (a.e.). Multiplying (2.23) by $\varphi_1(x)$, and integrating over D , we conclude that $\mu = \mu(k) > 0$. Then $\frac{u(x)}{\mu}$ is the solution of (2.21), satisfying (2.22) (a.e.). \diamond

We now study the *global solution curve* $\mu = \phi(\xi)$ for the problem (2.9), with $u(x) \in H^2(D) \cap H_0^1(D)$, in case $g(u)$ is either convex or concave.

Theorem 2.3 *Assume that $f(x) \in H^2(D)$, $f(x) > 0$ a.e., and $g(u) \in C^2(R)$ satisfies the condition (2.10), and we have $g'(u) \leq \nu_1 < \nu$ for all $u \in R$. Assume that either $g''(u) > 0$, or $g''(u) < 0$ holds for all $u \in R$. Then the solution curve of the problem (2.9) $\mu = \phi(\xi)$ is either monotone, or it has exactly one critical point, which is the point of global minimum in case $g''(u) > 0$ for all $u \in R$, and the point of global maximum in case $g''(u) < 0$ for all $u \in R$.*

Proof: By the Theorem 2.1, the problem (2.9) has a solution curve $(u, \mu)(\xi)$, where ξ is the generalized first harmonic of $u(\xi)$. Differentiate the equation (2.9) in ξ

$$(2.24) \quad \Delta u_\xi + g'(u)u_\xi = \mu'(\xi)f(x) \text{ in } D, \quad u_\xi = 0 \text{ on } \partial D.$$

We claim that $u_\xi(x) \not\equiv 0$ for all $\xi \in R$. Indeed, since $u(x) = \xi f(x) + U(x)$, we have $u_\xi(x) = f(x) + U_\xi(x)$. If $u_\xi(x) \equiv 0$, then $U_\xi(x) = -f(x)$, but $\int_D U_\xi(x)f(x) dx = 0$, a contradiction.

Assume that $\mu'(\xi_0) = 0$ at some ξ_0 . Denoting $w(x) = u_\xi$ at $\xi = \xi_0$, we see that $w(x)$ is a non-trivial solution of

$$(2.25) \quad \Delta w + g'(u)w = 0 \text{ in } D, \quad w = 0 \text{ on } \partial D.$$

Since $g'(u) < \lambda_2$, it follows that $w(x) > 0$ in D . In the spirit of [13] and [16], we differentiate the equation (2.24) once more in ξ , and set $\xi = \xi_0$:

$$(2.26) \quad \Delta u_{\xi\xi} + g'(u)u_{\xi\xi} + g''(u)w^2 = \mu''(\xi_0)f(x) \text{ in } D, \quad u_{\xi\xi} = 0 \text{ on } \partial D.$$

Combining the equations (2.25) and (2.26), we have

$$\mu''(\xi_0) \int_D w f(x) dx = \int_D g''(u)w^3 dx.$$

It follows that $\mu''(\xi_0) > 0$ ($\mu''(\xi_0) < 0$) in case $g''(u) > 0$ for all $u \in R$ ($g''(u) < 0$ for all $u \in R$), so that any critical point of $\mu(\xi)$ is a local minimum (maximum), and hence at most one critical point is possible. \diamond

It is now easy to classify all of the possibilities.

Theorem 2.4 Assume that $f(x) \in H^2(D)$, $f(x) > 0$ a.e., and $g(u) \in C^2(R)$ satisfies the condition (2.10), and we have $g'(u) \leq \nu_1 < \nu$ for all $u \in R$. Assume also that $g''(u) > 0$ for all $u \in R$.

(i) If $\gamma_1, \gamma_2 < \lambda_1$, then the problem (2.9) has a unique solution for any $\mu \in R$. Moreover, the solution curve $\mu = \phi(\xi)$ is defined, and monotone decreasing for all $\xi \in R$.

(ii) If $\lambda_1 < \gamma_1, \gamma_2 < \nu$, then the problem (2.9) has a unique solution for any $\mu \in R$. Moreover, the solution curve $\mu = \phi(\xi)$ is defined, and monotone increasing for all $\xi \in R$.

(iii) If $\gamma_1 < \lambda_1 < \gamma_2 < \nu$, then there is a critical μ_0 , so that the problem (2.9) has exactly two solutions for $\mu > \mu_0$, it has a unique solution at $\mu = \mu_0$, and no solutions for $\mu < \mu_0$. Moreover, the solution curve $\mu = \phi(\xi)$ is defined for all $\xi \in R$, it is parabola-like, and μ_0 is its global minimum value.

Proof: The convexity of $g(u)$ implies that $\gamma_1 < \gamma_2$. By the Theorem 2.3, the problem (2.9) has a solution curve $\mu = \phi(\xi)$, defined for all $\xi \in R$, which is either monotone, or it has exactly one critical point, which is the point of global minimum. Decompose $u(x) = \bar{\xi}\varphi_1(x) + \bar{U}(x)$, where $\int_D \bar{U}(x)\varphi_1(x) dx = 0$, and $\bar{\xi}$ is the first harmonic. We have

$$u(x) = \xi f(x) + U(x) = \bar{\xi}\varphi_1(x) + \bar{U}(x).$$

Multiplying this by $f(x)$, and integrating

$$\xi \int_D f^2(x) dx = \bar{\xi} \int_D f(x)\varphi_1(x) dx + \int_D \bar{U}(x)f(x) dx.$$

By the Corollary 1 of Lemma 2.4, $\int_D \bar{U}(x)f(x) dx$ is uniformly bounded. It follows that $\bar{\xi} \rightarrow \infty (-\infty)$ if and only if $\xi \rightarrow \infty (-\infty)$, providing us with a “bridge” between ξ and $\bar{\xi}$.

Multiply the equation in (2.9) by ϕ_1 , and integrate:

$$(2.27) \quad \mu \int_D f(x)\phi_1(x) dx = -\lambda_1 \bar{\xi} + \int_D g(u)\phi_1(x) dx,$$

with $\int_D f(x)\phi_1(x) dx > 0$. If $\bar{\xi} > 0$ and large, then $u(x) = \bar{\xi}\varphi_1(x) + \bar{U}(x) > 0$ a.e. in D , and we have $u(x) < 0$ a.e. in D if $\bar{\xi} < 0$ and $|\bar{\xi}|$ is large. By the condition (2.10),

$$\mu \sim (\gamma_2 - \lambda_1) \bar{\xi}, \quad \text{when } \bar{\xi} > 0 \text{ and large},$$

$$\mu \sim (\gamma_1 - \lambda_1) \bar{\xi}, \quad \text{when } \bar{\xi} < 0 \text{ and } |\bar{\xi}| \text{ is large}.$$

These formulas give us the behavior of $\mu = \phi(\xi)$, as $\xi \rightarrow \pm\infty$, and the theorem follows. \diamond

The following result is proved similarly (the concavity of $g(u)$ implies that $\gamma_2 < \gamma_1$).

Theorem 2.5 *Assume that $f(x) \in H^2(D)$, $f(x) > 0$ a.e., and $g(u) \in C^2(R)$ satisfies the condition (2.10), and we have $g'(u) \leq \nu_1 < \nu$ for all $u \in R$. Assume also that $g''(u) < 0$ for all $u \in R$.*

(i) If $\gamma_1, \gamma_2 < \lambda_1$, then the problem (2.9) has a unique solution for any $\mu \in R$. Moreover, the solution curve $\mu = \phi(\xi)$ is monotone decreasing for all $\xi \in R$.

(ii) If $\lambda_1 < \gamma_1, \gamma_2 < \nu$, then the problem (2.9) has a unique solution for any $\mu \in R$. Moreover, the solution curve $\mu = \phi(\xi)$ is monotone increasing for all $\xi \in R$.

(iii) If $\gamma_2 < \lambda_1 < \gamma_1 < \nu$, then there is a critical μ_0 , so that the problem (2.9) has exactly two solutions for $\mu < \mu_0$, it has a unique solution at $\mu = \mu_0$, and no solutions for $\mu > \mu_0$. Moreover, the solution curve $\mu = \phi(\xi)$ is parabola-like, and μ_0 is its global maximum value.

It appears that there is less interest in concave nonlinearities, compared with the convex ones. This may be due to the fact that if one considers *positive* solutions of

$$\Delta u + g(u) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

and $g(0) \geq 0$, then the case of concave $g(u)$ is easy, and the convex case is interesting. However, if $g(0) < 0$, the situation may be reversed even for positive solutions, see e.g., [10]. For sign-changing solutions, it seems that the convex and concave cases are of equal complexity.

Examining the proofs, we see that the Theorems 2.4 and 2.5 hold verbatim for the problem

$$\Delta u + g(u) = \mu f(x) + e(x) \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

with $e(x) \in H^2(D)$ satisfying $\int_D e(x)f(x) dx = 0$, giving a generalization of the classical results of A. Ambrosetti and G. Prodi [1], and of M.S. Berger and E. Podolak [4].

3 A population model with fishing

One usually begins population modeling with a logistic model

$$(3.1) \quad u'(t) = au(t) - bu^2(t).$$

Here $u(t)$ can be thought of as the number of fish in a lake at time t ; a and b are positive constants. When $u(t)$ is small, $u^2(t)$ is negligible, and the population grows exponentially, but after some time the growth rate decreases. Now suppose the lake occupies some region $D \subset R^n$, and $u = u(x, t)$, with $x \in D$. Suppose that fish diffuses around the lake, and the population is near zero at the banks. Assume also there is time-independent fishing, accounted by the term $\mu f(x)$, where $f(x)$ is a positive function, and μ is a parameter. Then the model is

$$u_t(x, t) = au(x, t) - bu^2(x, t) + \Delta u(x, t) - \mu f(x) \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

We shall consider its steady state $u = u(x)$, satisfying

$$(3.2) \quad \Delta u(x) + au(x) - bu^2(x) - \mu f(x) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

It is customary in the population modeling to look for positive solutions. However one does not expect the solutions of (3.2) to remain positive, when the parameter $\mu > 0$ is varied (since $f(x) > 0$). Therefore, we shall admit sign-changing solutions, with the interpretation that some re-stocking of fish is necessary when $u(x) < 0$ (which presumably occurs near the banks, i.e., ∂D), to avoid the algae growth or other negative consequences. However, there is no reason to use the logistic model (3.1) for sign-changing u . When $u < 0$, it is still reasonable to assume that $u'(t) \approx au(t) < 0$, which corresponds to the assumption that the situation further deteriorates without re-stocking, but there seems to be no justification for the $-bu^2$ term.

We consider the following model ($f(x) > 0$)

$$(3.3) \quad \Delta u(x) + g(u(x)) - \mu f(x) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

where $g(u)$ is an extension of the logistic model to $u < 0$, which we describe next. Namely, we assume that $g(u) \in C^2(R)$, and it satisfies

$$(3.4) \quad g(u) = au - bu^2 \text{ for } u \geq 0, \text{ with } \lambda_1 < a < \nu, \text{ and } b > 0,$$

$$(3.5) \quad g'(u) < \nu, \text{ and } g''(u) < 0 \text{ for } u \in R,$$

where $\lambda_1 < \nu \leq \lambda_2$ was defined in Lemma 2.1. Our conditions imply that $g(u) \sim cu + d$ as $u \rightarrow -\infty$, for some constants $0 < c < \nu$, and $d > 0$.

When $\mu = 0$ (no fishing), the problem (3.3) has the trivial solution $u(x) \equiv 0$, and a unique positive solution $u_0(x)$, see e.g., P. Korman and A. Leung [12]. When $\mu > 0$ is varied these two solutions turn out to be

connected by a smooth solution curve. To prove this result, we shall need the following consequence of Lemma 3.3 in [2].

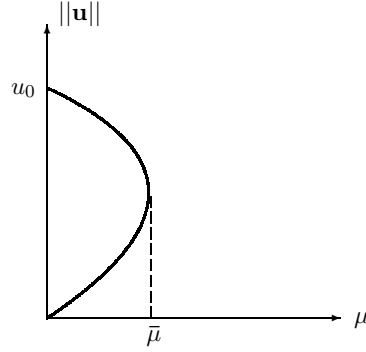


Figure 1. The solution curve for the fishing model

Lemma 3.1 *Let $u(x, \mu)$ denote the classical solution of (3.3), depending on a parameter $\mu \in \mathbb{R}$. Assume that $\mu_2 > \mu_1$, and $u(x, \mu_1) > 0$, $u(x, \mu_2) > 0$ for all $x \in D$. Then $u(x, \mu_2) > u(x, \mu_1)$ for all $x \in D$.*

Let $\xi_0 = \int_D u_0(x) f(x) dx > 0$ denote the first generalized harmonic of $u_0(x)$. We have the following result, for possibly sign-changing solutions.

Theorem 3.1 *Assume that the conditions (3.4) and (3.5) hold, and $f(x) \in C^\alpha(D)$, $f(x) > 0$ in D , $\alpha > 0$. Then in the (ξ, μ) plane there is a smooth parabola-like solution curve $\mu = \varphi(\xi)$ of (3.3), connecting the points $(0, 0)$ and $(\xi_0, 0)$. It has a unique point of maximum at some $\bar{\xi} \in (0, \xi_0)$, with $\bar{\mu} = \varphi(\bar{\xi}) > 0$. Equivalently, for $\mu \in [0, \bar{\mu})$ the problem (3.3) has exactly two solutions, it has exactly one solution at $\mu = \bar{\mu}$, and no solutions for $\mu > \bar{\mu}$. Moreover, all solutions lie on a parabola-like solution curve in the $(\mu, ||u||)$ plane, with a turn to the left.*

Proof: By Theorem 2.1, we continue the solution curve from the point $(\xi_0, 0)$ in the (ξ, μ) plane for decreasing ξ . By Lemma 3.1, it follows that $\mu > 0$ for ξ near ξ_0 , and $\xi < \xi_0$ (if $\mu < 0$, then $\xi > \xi_0$). By Theorem 2.5, this curve $\mu = \mu(\xi)$ has a unique critical point on $(0, \xi_0)$, which is a point of global maximum, and this curve links up to the point $(0, 0)$. This implies that the solution curve in the $(\mu, ||u||)$ plane is as in Figure 1, concluding the proof. \diamond

S. Oruganti et al [15] considered positive solutions of (3.2). They proved a similar result (as in Figure 1) for a sufficiently close to λ_1 . In that case, ξ_0 is small, and the entire solution curve is close to the point $(0, 0)$. Working with positive solutions only narrows the class of solutions considerably, and the result of [15] is probably the best one can get (for the picture as in Figure 1). We showed in [11] that the picture is different when $a > \lambda_2$ (in case of positive solutions).

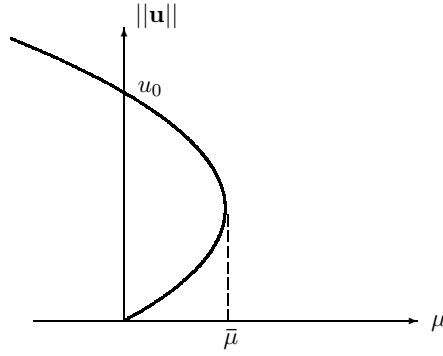


Figure 2. The fishing model with stocking of fish (when $\mu < 0$)

We show next that the upper branch of the solution curve in Theorem 3.1 continues for $\xi \in (-\infty, 0)$, with $\mu = \varphi(\xi)$ monotone decreasing, and $u(x) > 0$ in D . Moreover, $\lim_{\xi \rightarrow -\infty} \varphi(\xi) = +\infty$, implying that the solution curve in the $(\mu, ||u||)$ plane is as in Figure 2. Indeed, let us return to the solution point $(\xi_0, 0)$ in the (ξ, μ) plane. For $\xi > \xi_0$, we have $\mu < 0$ by Lemma 3.1. Then $u(x) > 0$, by the minimum principle, so that $g'(u) = a - 2u < \nu$, and Theorem 2.5 applies. One can interpret $\mu < 0$ as *stocking of fish*.

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